# IB Mathematics HL 12
# A Review of the Derivative

## 1 The Derivative at a Point

Consider a function $f$, and a point $a$ in the domain of $f$.[1] If we were to graph $y = f(x)$, we would find that the point $(a, f(a))$ lies on the graph. Similarly, for any $b$ in the domain of $f$, we could produce another point on the graph, $(b, f(b))$. We would then be able to construct the line passing through $(a, f(a))$ and $(b, f(b))$, after which we could examine the gradient of that line (a line passing through (at least) two points on the graph of a function is called a *secant* line).

This is essentially the idea that leads us to the definition of the derivative at a point $a$: we consider how the gradient of the secant line would change as the value of $b$ approaches the value of $a$. The gradient of the secant line would be given by

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

and $b$ approaching $a$ is of course one way of talking about a *limit*, leading us to consider

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a} \tag{1}$$

Equation (1) is one definition of the derivative of $f$ at a point $a$.

In our lessons, we've used an alternative, equivalent definition. Instead of using $b$ to represent a value in the domain of $f$, we've used $a + h$ instead. Our second point on the graph of the function then has coordinates $(a + h, f(a + h))$, and so the secant line would have gradient

$$\frac{\Delta y}{\Delta x} = \frac{f(\color{red}{a+h}) - f(a)}{(\color{red}{a+h}) - a}, \text{ or more simply,}$$
$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

The fraction appearing on the right above is sometimes referred to as the *Newton quotient*. Now, instead of letting $b$ approach $a$, we can let $h$ approach 0 to get the same effect. So, we're left to consider

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

Equation (2) is the definition of the derivative of $f$ at a point $a$ that we used in class.

[1] You may find that it is helpful to consider a particular function and a particular value for $a$, and I'd recommend thinking of $f(x) = x^2$ and $a = 1$ if you get stuck in any part of the discussion that follows.

If we make use of the Lagrange notation for the derivative of $f$, then the definition of the derivative of $f$ at a point $a$ is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{3}$$

Note that, for a particular function $f$ and a particular value $a$, the derivative of $f$ at $a$ will be a *numerical value*. In class we saw that, for $f(x) = x^2$ and $a = 1$, we get $f'(1) = 2$, and similarly we found that $f'(2) = 4$ and $f'(3) = 6$.

### 1.1 Questions

1. Use the definition of the derivative to find the derivative of $f(x) = -x^2 + 4x$ at $x = 5$.

2. Consider the function $f(x) = 2x$.

   (a) Use the definition of the derivative to show that the derivative of this function at $x = 3$ is 2.

   (b) Prove that $f'(a) = 2$ for any $a \in \mathbb{R}$.

## 2 *The* Derivative

After calculating the derivative of a simple function like $f(x) = x^2$ for various values of $a$, you may start to observe a pattern that would allow you to predict the value of the derivative without calculating the relevant limit. If you were interested in finding the value of $f'(a)$ for several different values of $a$, you might instead hope to find a *function* that would, given the value of $a$, allow you to calculate the value of $f'(a)$ without your needing to evaluate a limit. This leads to the notion of *the* derivative, which is exactly the function that gives you those values directly.

**Definition** (The Derivative)**.** Given a differentiable function $f$, the derivative of $f$, represented by $f'$, is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

A function that has a derivative at all points in its domain is called *differentiable*. While we won't pause to consider such functions now, some functions are not differentiable: they may fail to be differentiable at certain points in their domain, or they may be nowhere differentiable.

### 2.1 Derivatives of Polynomials

When we need to determine the derivative of a given function, we would usually like to avoid having to determine the derivative by using the definition (and so, explicitly considering a limit). Instead, we'd like to develop techniques for determining the derivative by easier methods, when possible. To this end, some of you will already be familiar with the following result.

**Theorem** (The Derivative of $f(x) = ax^n$)**.** *Given a function of the form $f(x) = ax^n$ for some constant $a \in \mathbb{R}$, and $n \in \mathbb{Z}^+$, we have*

$$f'(x) = anx^{n-1}$$

Of course, this result alone doesn't allow us to find the derivative of *any* polynomial, but only those like $f(x) = 3x^2$, with derivative $f'(x) = 6x$, and $f(x) = \frac{2}{5}x^5$, with derivative $f'(x) = 2x^4$. Fortunately, we can easily extend this using the following result (which is rather simple to prove using the definition of the derivative and the relevant properties of limits).

**Theorem** (The Additive Property of Derivatives)**.** *Given differentiable functions $f$ and $g$, the function $h$ defined by $h(x) = f(x) + g(x)$ is such that*

$$h'(x) = f'(x) + g'(x)$$

In other words, *the derivative of a sum is the sum of the derivatives.*

Now, using this result, we *can* find the derivative of any polynomial.[2] For example, if $f(x) = 5x^4 - 2x^3 + 7x - 3$, then $f'(x) = 20x^3 - 6x^2 + 7$.

In fact, we have another powerful result at our disposal, as the earlier result concerning the derivative of functions of the form $f(x) = ax^n$ was unnecessarily restricted: we can, in fact, allow $n$ to be *any nonzero real number*. Consider, for example the function $f(x) = \sqrt{x}$. We can reason as follows.

$$\begin{aligned} f(x) &= \sqrt{x} \\ &= x^{\frac{1}{2}}, \text{ and then, taking } n = \frac{1}{2} \text{ in our earlier theorem, we get} \\ f'(x) &= \frac{1}{2}x^{-\frac{1}{2}}, \text{ which could be rewritten as} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Note that (both for the derivative and the original function), the domain must be restricted to non-negative real numbers.

The result above involved a positive rational exponent, but negative exponents are also permitted. Thus, for example, if $f(x) = \frac{1}{3x}$, then $f'(x) = -\frac{1}{3x^2}$. Many of the important results we'll cover later in the course concern how we can find derivatives for increasingly complicated functions without having to return to the limit definition of the derivative.

## 2.2 Questions

1. Find the derivative of $f(x) = -2x^3$

   (a) using the technique just introduced for finding derivatives of polynomials.

[2] Strictly speaking we will also be using the result that the derivative of a constant is 0, another result that is quite straightforward to prove.

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(b) using the definition of the derivative.

2. Use the method introduced in this section to show that, if $f(x) = \frac{1}{3x}$, then $f'(x) = -\frac{1}{3x^2}$.

3. Find the derivative of $f(x) = x^4 - \frac{1}{2x^2} + \sqrt{x}$.

# 3 So What Exactly *is* a Derivative?

We can easily find derivatives for a wide variety of functions, but what exactly *is* a derivative? The short (but not very informative) reply is that the derivative of a function is another function, namely, the one given by taking the limit of the Newton quotient (as defined at the beginning of Section 2). Of course, this probably isn't very helpful, and to get a better understanding of derivatives we'll need to consider some of the ways in which we can interpret the derivative.

Consider the gradient of a line between two points $(x_1, y_1)$ and $(x_2, y_2)$. You know that the gradient of the line passing through those two points is given by

$$\frac{y_2 - y_1}{x_2 - x_1}$$

That is, after all, the definition of the gradient. However, you also know that the gradient of a line gives information about its steepness, you know that the line given by $y = 5x - 2$ is steeper than the line given by $y = 2x + 14$, that lines with positive gradients are increasing, lines with negative gradients are decreasing, and you know that you can use any two distinct points on the same line to calculate the gradient and you'll get the same value. These results are obvious to you now, but probably aren't obvious to someone who has just been given the definition of the gradient. We are in a similar situation now with the derivative: we've seen the definition and we can calculate derivatives for a variety of different functions, but we haven't yet discussed what information the derivative provides.

## 3.1 Derivatives and Tangent Lines

Part of what makes the derivative so useful is the relationship between tangent lines and the derivative. A line that is *tangent* to a curve (in other words, a *tangent line*) is one that, roughly speaking, just touches the curve at a single point.[3]

The important connection between tangent lines and derivatives is expressed below.

**Definition** (Derivatives and Tangent Lines)**.** Given a differentiable function $f$ and a value $a$ in the domain of $f$, the gradient of the line tangent to the graph of $y = f(x)$, passing through the point $(a, f(a))$, is equal to $f'(a)$.

[3]This description of a tangent line is not entirely accurate, as you'll see when working through the questions at the end of this section.

In other words, *derivatives give gradients of tangent lines*.[4]

This connection between derivatives and tangent lines can be illustrated by considering the function $f(x) = x^2$. If we graph $y = f(x)$, we get a parabola. Since $f'(x) = 2x$, we can easily calculate that $f'(1) = 2$, $f'(2) = 4$, and $f'(-1) = -2$. What do these values tell us? Well, the points with coordinates $(1,1)$ (with $x$-coordinate 1), $(2,4)$ (with $x$-coordinate 2), and $(-1,1)$ (with $x$-coordinate $-1$), all lie on the parabola, and for each point there is a a line tangent to the parabola that passes through that point. Calculating the derivatives tells us that

- the tangent line passing through $(1,1)$ has gradient 2, since $f'(1) = 2$,
- the tangent line passing through $(2,4)$ has gradient 4, since $f'(2) = 4$, and
- the tangent line passing through $(-1,1)$ has gradient $-2$, since $f'(-1) = -2$.

Notice that the calculation of the gradient of the tangent line only involves the $x$-coordinate of the relevant point on the curve: we can determine, for example, that the tangent line to the curve $y = x^2$ that passes through the point with $x$-coordinate 2.5 is 5, without having to calculate the $y$-coordinate of the point on the curve.

With a bit more work, we can also find the *equations* of the tangent lines (and here we do need to find the $y$-coordinates of the points on the curve). If we are trying to find the equations of each line in the form $y = mx + c$, then we can use the gradient (the value of $m$) and the given point on the parabola to determine the value of $c$ in each case. For example, the equation of the tangent to the curve $y = x^2$ passing through $(1,1)$ is $y = 2x - 1$.

### 3.1.1 Questions

1. Find the equation of the tangent line to the curve given by $y = x^2$ at the point on the curve with $x$-coordinate 2.5.
2. Consider the function $f(x) = x^3 - 2x^2 - x + 2$.
   (a) Find the equation of the tangent line at the point on the curve $y = f(x)$ with $x = 0$.
   (b) Plot the graph of both the original function $f$ and the tangent line from part (a) on the same set of axes.
   (c) Does the tangent line found in part (a) intersect the original curve at only one point?

[4]Note that this 'result' does not simply note a connection between tangent lines and derivatives, it actually gives the precise definition of a tangent line: the tangent line to the curve defined by $y = f(x)$ at a point $(a, f(a))$ is the the line passing through that point with gradient $f'(a)$. Thus, the gradient of the tangent line is *defined* to be equal to the value of the derivative. This allows us to avoid issues with the 'touches the curve at only one point' description of the tangent line—see question 2c in 3.1.1.

## 3.2 Derivatives and Rates of Change

Calculus is sometimes described as the study of *rates of change*, and the derivative at a point can also be interpreted as an *instantaneous rate of change*. To understand what that means, consider the following example.

A car has been driving for 30 minutes, and has covered a distance of 40 km. If you were asked to calculate the *average* speed of the car for this half-hour journey, you would calculate

$$\frac{40 \text{ km}}{0.5 \text{ h}} = 80 \text{ km/h}$$

Of course, over the 30 minute journey you might expect that the car was occasionally travelling faster than, and occasionally travelling slower than, 80 km/h. However, as far as the *average* speed is concerned, the calculation above is correct: you divide the total distance travelled by the total time taken to get the average speed over that interval—it doesn't matter whether the speed of the car was constant or not.

Let's now consider a similar scenario. Let's assume that the car *is* travelling at a constant speed of 80 km/h down a long, straight track. Along the track we've placed a marker, and we will start a stopwatch when the car passes that marker and then monitor the distance travelled as the car continues along the track. We could then plot a distance-time graph for this situation, as shown in Figure 1. Now consider: how could the speed of the car be determined from this graph? Given that *speed* is the *rate of change of distance with respect to time*, you should recognise that the speed of the car would be represented by *the gradient of the line*, and so calculating the gradient of the line in the distance-time graph would reveal the speed of the car.

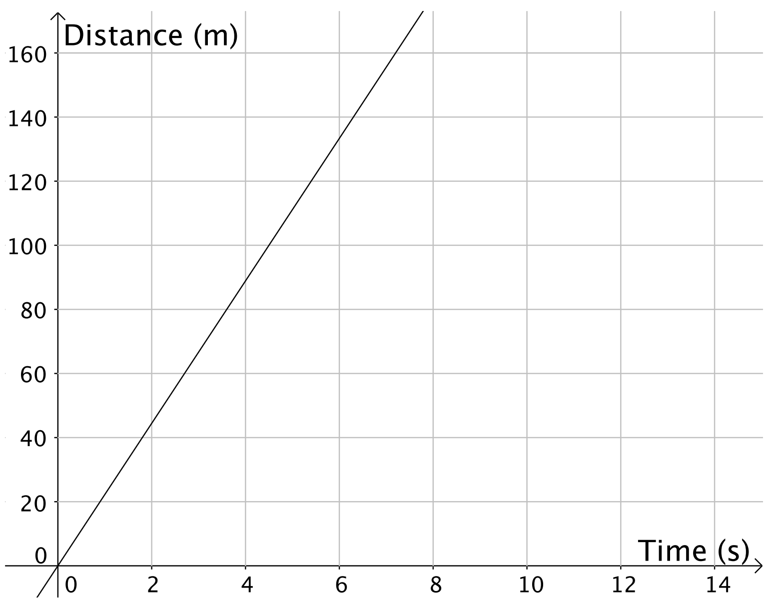


Figure 1: Distance-Time Graph for a Car Travelling at Constant Speed

### 3.2.1 Accelerated Motion

If cars always travelled at a constant speed we would be able to end our discussion here, but out in the 'real world' cars are constantly changing speed. A car that starts from rest has an initial speed of 0 km/h, but several seconds later that car could very well be travelling at a speed of 80 km/h. If the car is changing speed—if it is accelerating or decelerating—then the distance-time graph will not be a straight line, and so we will not be able to determine the speed of the car by finding a gradient as above.

Consider, then, a more realistic scenario, in which a car starts from rest and accelerates to 80 km/h in 10 seconds, with a distance time graph as shown in Figure 2.

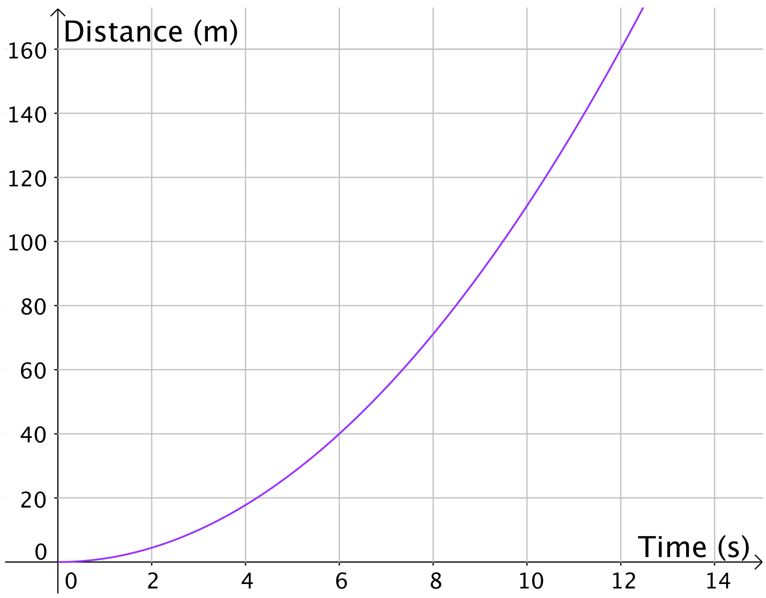


Figure 2: Distance-Time Graph for an Accelerating Car

Here, the function that gives the distance travelled by the car after $x$ seconds is

$$f(x) = \left(\frac{10}{9}\right)x^2$$

Using this function, we can easily verify that the point $(6, 40)$ lies on the curve. What was the car's average speed over the first 6 seconds? This is easily shown to be 24 km/h (since the car has travelled 40 m in 6 seconds), and corresponds to the gradient of the line shown in Figure 3.

Similarly, you can calculate that the car's average speed over the next 6 seconds is 72 km/h, which corresponds to the gradient of the line shown in Figure 4.

What none of these calculations provides, though, is the actual speed of the car at *exactly* 6 seconds—if you were to look at the car's speedometer 6 seconds after it started to move, what would you see? You might expect that it would be indicate a speed greater than 24 km/h, but less than 72 km/h. Given what we know about the

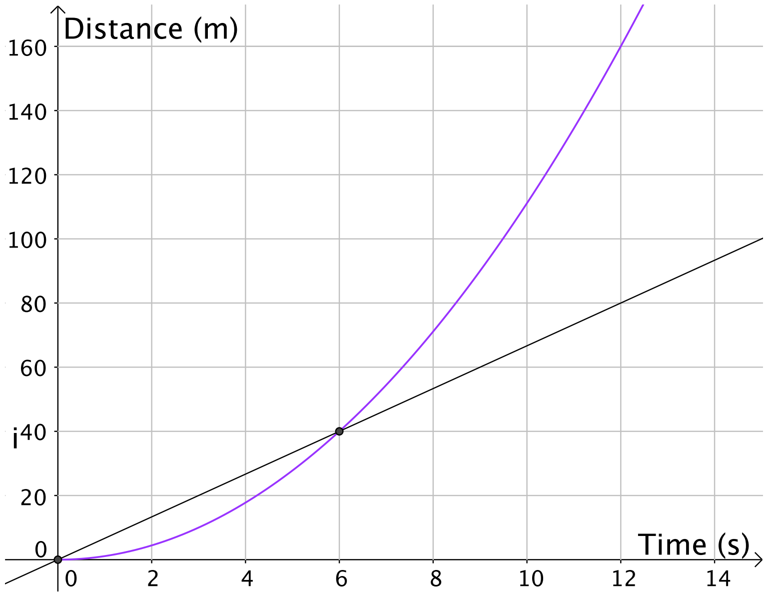


Figure 3: The average speed of the car in the first 6 seconds.

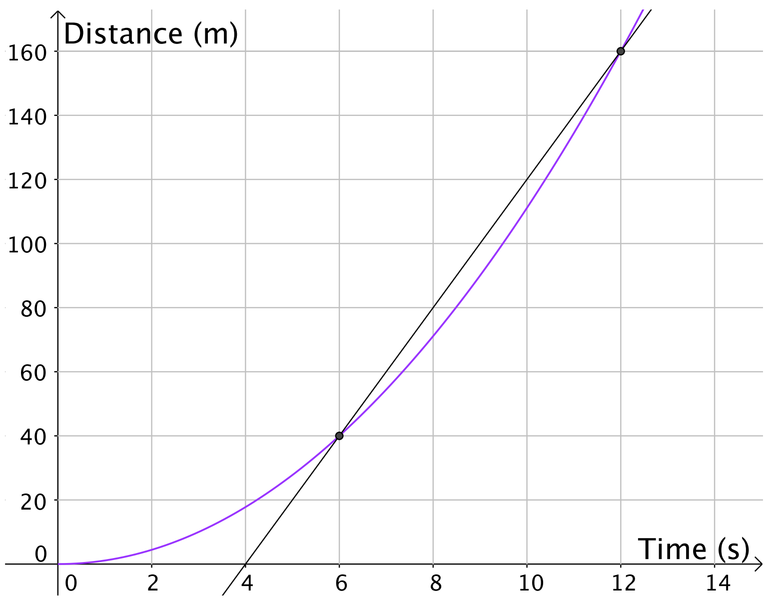


Figure 4: The average speed of the car in the next 6 seconds.

distance-time graph for the car, is there some way we can calculate its actual speed at exactly 6 seconds?

Knowing what you now know about the derivative, you may suspect that it can be used to answer the question above. Since $f'(x) = \left(\frac{20}{9}\right)x$, we have $f'(6) = \frac{40}{3}$. As you know, this value corresponds to the gradient of the tangent line passing through the point $(6,40)$, as shown in Figure 5.

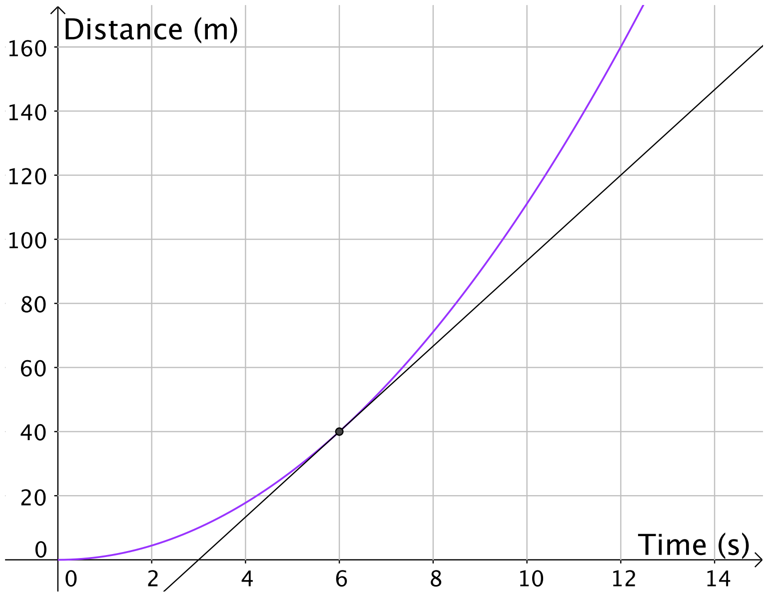


Figure 5: The speed of the car at 6 seconds.

The gradient of the tangent line reflects the units of measurement used in the graph, and so the gradient of the tangent line is a value whose units are metres per second. Converting to km/h, we get that the gradient is equivalent to 48 km/h. If you were to look at the car's speedometer exactly 6 seconds after it began to accelerate, you would see that it was travelling at exactly 48 km/h. A moment later it would be travelling at a speed faster than 48 km/h, and a moment earlier it would have been travelling at a speed less than 48 km/h. Thus, since the car is accelerating, its speed is exactly 48 km/h for only an instant, and it is in this sense that the derivative gives us an *instantaneous rate of change*.

#### 3.2.2 Questions

All questions below concern the accelerating car from Section 3.2.1.

1. (a) Show that the car is travelling at 80 km/h 10 seconds after it begins to move.
   (b) Calculate the average speed of the car over the first 10 seconds.
   (c) Find the time at which the car is travelling at exactly the speed you calculated in question 1b.

2. (a) Find the derivative of the function

$$f'(x) = \left(\frac{20}{9}\right)x$$

   This new function is a called *second* derivative.

(b) Explain why the units of the second derivative are $ms^{-2}$. What is the acceleration of the car?