IB Mathematics HL 13
Complex Numbers: Question 14 Notes

14. b) The complex number z is defined by $z = \cos\theta + i\sin\theta$.

ii) Deduce that $z^n + z^{-n} = 2\cos n\theta$.

Note that the expression on the left is the sum of a complex number and its conjugate, hence the result is a *real number*. Since it's a real number, the argument of $z^n + z^{-n}$ will either be 0 (if $z^n + z^{-n}$ is positive) or π (if $z^n + z^{-n}$ is negative). In general, then, the argument of the real number $2\cos n\theta$ is *not* $n\theta$.

c) i) Find the binomial expansion of $\left(z + z^{-1}\right)^5$.

$$\begin{aligned}\left(z+z^{-1}\right)^5 &= z^5 + \binom{5}{1}z^4\left(z^{-1}\right)^1 + \binom{5}{2}z^3\left(z^{-1}\right)^2 + \binom{5}{3}z^2\left(z^{-1}\right)^3 + \binom{5}{4}z\left(z^{-1}\right)^4 + \left(z^{-1}\right)^5 \\ &= z^5 + 5z^4z^{-1} + 10z^3z^{-2} + 10z^2z^{-3} + 5zz^{-4} + z^{-5} \\ &= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5} \\ &= \left(z^5 + z^{-5}\right) + 5\left(z^3 + z^{-3}\right) + 10\left(z + z^{-1}\right)\end{aligned}$$

ii) Hence, show that $\cos^5\theta = \frac{1}{16}(a\cos 5\theta + b\cos 3\theta + c\cos\theta)$, where $a, b,$ and c are positive integers to be found.

From b) ii) we get that $z + z^{-1} = 2\cos\theta$, and so

$$\begin{aligned}\left(z+z^{-1}\right)^5 &= (2\cos\theta)^5 \\ &= 32\cos^5\theta \qquad (1)\end{aligned}$$

From c) i), we get *another* way to expand $\left(z+z^{-1}\right)^5$,

$$\begin{aligned}\left(z+z^{-1}\right)^5 &= \left(z^5 + z^{-5}\right) + 5\left(z^3 + z^{-3}\right) + 10\left(z + z^{-1}\right) \quad \text{using b) ii) again gives} \\ &= 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta) \\ &= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta \qquad (2)\end{aligned}$$

Finally, combining (1) and (2) gives

$$\begin{aligned}32\cos^5\theta &= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta \\ \cos^5\theta &= \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)\end{aligned}$$

So, $a = 1, b = 5,$ and $c = 10$.