IB Mathematics HL 12
Polynomials Challenge

1. Consider a polynomial equation of the form $ax^2 + bx + c = 0$.

   (a) Complete the square to express the left side in vertex form.

   (b) Rearrange your equation from part (a) to isolate x. Name the formula you've just derived.

2. Let $\mathbb{R}[x]$ represent the set of polynomials with real coefficients, and $\mathbb{C}[x]$ represent the set of polynomials with complex coefficients.[1]

   A polynomial $p(x) \in \mathbb{R}[x]$ is said to be *irreducible* in $\mathbb{R}[x]$ if it cannot be written as a product of non-constant polynomials of lesser degree. So, $x^2 - 5$ is reducible, as it is equal to $\left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right)$, but 254, $5x - 7$, and $x^2 + 1$ (of degrees 0, 1, and 2, respectively) are all irreducible in $\mathbb{R}[x]$.

   If we consider instead $\mathbb{C}[x]$, the only irreducible polynomials are of degree < 2 (this is a consequence of the Factor Theorem and the Fundamental Theorem of Algebra). Here, for example, $x^2 + 1$ *does* factor as $(x - \mathrm{i})(x + \mathrm{i})$.

   (a) Show that, for any complex number $z = a + b\mathrm{i}$,

   $$(x - z)(x - z^*)$$

   is an element of $\mathbb{R}[x]$.

   Note that, for any two elements $p(x)$ and $q(x)$ of $\mathbb{R}[x]$, the product $p(x)q(x)$ is also an element of $\mathbb{R}[x]$.

   (b) Using your results from part (a), explain why there are no irreducible polynomials of degree > 2 in $\mathbb{R}[x]$.

[1] Note that $\mathbb{R}[x] \subset \mathbb{C}[x]$.