Exponents and Logarithms Review *[63 marks]*

A city is concerned about pollution, and decides to look at the number of people using taxis. At the end of the year 2000, there were 280 taxis in the city. After *n* years the number of taxis, *T*, in the city is given by

$$T = 280 \times 1.12^n.$$

1a. (i) Find the number of taxis in the city at the end of 2005. *[6 marks]*

(ii) Find the year in which the number of taxis is double the number of taxis there were at the end of 2000.

Markscheme

(i)
$n = 5$ ***(A1)***

$T = 280 \times 1.12^5$

$T = 493$ ***A1 N2***

(ii) evidence of doubling ***(A1)***

e.g. 560

setting up equation ***A1***

e.g.
$280 \times 1.12^n = 560$,
$1.12^n = 2$

$n = 6.116\ldots$ ***(A1)***

in the year 2007 ***A1 N3***

[6 marks]

Examiners report

A number of candidates found this question very accessible. In part (a), many correctly solved for *n*, but often incorrectly answered with the year 2006, thus misinterpreting that 6.12 years after the end of 2000 is in the year 2007.

1b. At the end of 2000 there were 25600 people in the city who used taxis. *[6 marks]*

After *n* years the number of people, *P*, in the city who used taxis is given by

$$P = \frac{2560000}{10 + 90e^{-0.1n}}.$$

(i) Find the value of *P* at the end of 2005, giving your answer to the nearest whole number.

(ii) After seven complete years, will the value of *P* be double its value at the end of 2000? Justify your answer.

Markscheme

(i)
$P = \frac{2560000}{10+90e^{-0.1(5)}}$ ***(A1)***

$P = 39635.993\ldots$ ***(A1)***

$P = 39636$ ***A1*** ***N3***

(ii)
$P = \frac{2560000}{10+90e^{-0.1(7)}}$

$P = 46806.997\ldots$ ***A1***

not doubled ***A1*** ***N0***

valid reason for **their** answer ***R1***

e.g.
$P < 51200$

[6 marks]

Examiners report

Many found correct values in part (b) and often justified their result by simply noting the value after seven years is less than 51200. A common alternative was to divide 46807 by 25600 and note that this ratio is less than two. There were still a good number of candidates who failed to provide any justification as instructed.

1c. Let *R* be the ratio of the number of people using taxis in the city to the number of taxis. The city will reduce the number of taxis if
$R < 70$. *[5 marks]*

(i) Find the value of *R* at the end of 2000.

(ii) After how many complete years will the city first reduce the number of taxis?

Markscheme

(i) correct value ***A2*** ***N2***

e.g.
$\frac{25600}{280}$, 91.4,
$640 : 7$

(ii) setting up an inequality (accept an equation, or reversed inequality) ***M1***

e.g.
$\frac{P}{T} < 70$,
$\frac{2560000}{(10+90e^{-0.1n})280\times1.12^{n}} < 70$

finding the value
$9.31\ldots$ ***(A1)***

after 10 years ***A1*** ***N2***

[5 marks]

Examiners report

Part (c) proved more challenging to candidates. Many found the correct ratio for *R*, however few candidates then created a proper equation or inequality by dividing the function for *P* by the function for *T* and setting this equal (or less) than 70. Such a function, although unfamiliar, can be solved using the graphing or solving features of the GDC. Many candidates chose a tabular approach but often only wrote down one value of the table, such as
$n = 10$,
$R = 68.3$. What is essential is to include the two values between which the correct answer falls.Sufficient evidence would include
$n = 9$,
$R = 70.8$ so that it is clear the value of
$R = 70$ has been surpassed.

Let
$f(x) = \mathrm{e}^{x+3}$.

2a. (i) Show that
$f^{-1}(x) = \ln x - 3$. *[3 marks]*

(ii) Write down the domain of
f^{-1} .

Markscheme

(i) interchanging *x* and *y* (seen anywhere) ***M1***

e.g.
$x = \mathrm{e}^{y+3}$

correct manipulation ***A1***

e.g.
$\ln x = y + 3$,
$\ln y = x + 3$

$f^{-1}(x) = \ln x - 3$ ***AG*** ***N0***

(ii)
$x > 0$ ***A1*** ***N1***

[3 marks]

Examiners report

Many candidates interchanged the
x and
y to find the inverse function, but very few could write down the correct domain of the inverse, often giving
$x \geq 0$,
$x > 3$ and "all real numbers" as responses.

2b. Solve the equation
$f^{-1}(x) = \ln \frac{1}{x}$. *[4 marks]*

Markscheme

collecting like terms; using laws of logs ***(A1)(A1)***

e.g.
$\ln x - \ln\left(\frac{1}{x}\right) = 3$,
$\ln x + \ln x = 3$,
$\ln\left(\frac{x}{\frac{1}{x}}\right) = 3$,
$\ln x^2 = 3$

simplify ***(A1)***

e.g.
$\ln x = \frac{3}{2}$,
$x^2 = \mathrm{e}^3$

$x = \mathrm{e}^{\frac{3}{2}}\left(= \sqrt{\mathrm{e}^3}\right)$ ***A1 N2***

[4 marks]

Examiners report

Where students attempted to solve the equation in (b), most treated $\ln x - 3$ as $\ln(x-3)$ and created an incorrect equation from the outset. The few who applied laws of logarithms often carried the algebra through to completion.

A population of rare birds, P_t, can be modelled by the equation $P_t = P_0\mathrm{e}^{kt}$, where P_0 is the initial population, and t is measured in decades. After one decade, it is estimated that $\frac{P_1}{P_0} = 0.9$.

3a. (i) Find the value of k. *[3 marks]*

(ii) Interpret the meaning of the value of k.

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Markscheme

(i) valid approach ***(M1)***

eg $0.9 = \mathrm{e}^{k(1)}$

$k = -0.105360$

$k = \ln 0.9\ (\text{exact}),\ -0.105$ ***A1 N2***

(ii) correct interpretation ***R1 N1***

eg population is decreasing, growth rate is negative

[3 marks]

Examiners report

Part (a) was generally done well, with many candidates able to find the value of k correctly and to interpret its meaning. Lack of accuracy was occasionally a concern, with some candidates writing their value of k to 2 significant figures or evaluating $\ln(0.9)$ incorrectly.

3b. Find the least number of **whole** years for which $\frac{P_t}{P_0} < 0.75$. *[5 marks]*

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Markscheme

METHOD 1

valid approach (accept an equality, but do not accept 0.74) ***(M1)***

eg $P < 0.75P_0,\ P_0\mathrm{e}^{kt} < 0.75P_0,\ 0.75 = \mathrm{e}^{t\ln 0.9}$

valid approach to solve **their** inequality ***(M1)***

eg logs, graph

$t > 2.73045\ (\text{accept } t = 2.73045)\ (2.73982 \text{ from } -0.105)$ ***A1***

28 years ***A2*** ***N2***

METHOD 2

valid approach which gives both crossover values accurate to at least 2 sf ***A2***

eg $\frac{P_{2.7}}{P_0} = 0.75241\ldots,\ \frac{P_{2.8}}{P_0} = 0.74452\ldots$

$t = 2.8$ ***(A1)***

28 years ***A2*** ***N2***

[5 marks]

Examiners report

Few candidates were successful in part (b) with many unable to set up an inequality or equation which would allow them to find the condition on t. Some were able to find the value of t in decades but most were unable to correctly interpret their inequality in terms of the least number of whole years. While a solution through analytic methods was readily available, very few students attempted to use their GDC to solve their initial equation or inequality.

4a. Find the value of $\log_2 40 - \log_2 5$. *[3 marks]*

Markscheme

evidence of correct formula ***(M1)***

eg
$\log a - \log b = \log \frac{a}{b}$,
$\log\left(\frac{40}{5}\right)$,
$\log 8 + \log 5 - \log 5$

Note: Ignore missing or incorrect base.

correct working ***(A1)***

eg
$\log_2 8$,
$2^3 = 8$

$\log_2 40 - \log_2 5 = 3$ ***A1 N2***

[3 marks]

Examiners report

Many candidates readily earned marks in part (a). Some interpreted
$\log_2 40 - \log_2 5$ to mean
$\frac{\log_2 40}{\log_2 5}$, an error which led to no further marks. Others left the answer as
$\log_2 5$ where an integer answer is expected.

4b. Find the value of
$8^{\log_2 5}$. *[4 marks]*

Markscheme

attempt to write
8 as a power of
2 (seen anywhere) ***(M1)***

eg
$(2^3)^{\log_2 5}$,
$2^3 = 8$,
2^a

multiplying powers ***(M1)***

eg
$2^{3\log_2 5}$,
$a\log_2 5$

correct working ***(A1)***

eg
$2^{\log_2 125}$,
$\log_2 5^3$,
$\left(2^{\log_2 5}\right)^3$

$8^{\log_2 5} = 125$ ***A1 N3***

[4 marks]

Examiners report

Part (b) proved challenging for most candidates, with few recognizing that changing 8 to base 2 is a helpful move. Some made it as far as $2^{3\log_2 5}$ yet could not make that final leap to an integer.

Jose takes medication. After *t* minutes, the concentration of medication left in his bloodstream is given by $A(t) = 10(0.5)^{0.014t}$, where *A* is in milligrams per litre.

5a. Write down $A(0)$. *[1 mark]*

Markscheme

$A(0) = 10$ ***A1 N1***

[1 mark]

Examiners report

For a later question in Section A, a pleasing number of candidates made good progress. Some candidates believed that raising a base to the zero power gave zero which indicated that they most likely did not begin by analysing the function with their GDC. For part (c), many candidates could set up the equation correctly and had some idea to apply logarithms but became lost in the algebra. Those who used their GDC to find when the function equalled 0.395 typically did so successfully. A common error for those who obtained a correct value for time in minutes was to treat 5.55 hours as 5 hours and 55 minutes after 13:00.

5b. Find the concentration of medication left in his bloodstream after 50 minutes. *[2 marks]*

Markscheme

substitution into formula ***(A1)***

e.g. $10(0.5)^{0.014(50)}$, $A(50)$

$A(50) = 6.16$ ***A1 N2***

[2 marks]

Examiners report

For a later question in Section A, a pleasing number of candidates made good progress. Some candidates believed that raising a base to the zero power gave zero which indicated that they most likely did not begin by analysing the function with their GDC. For part (c), many candidates could set up the equation correctly and had some idea to apply logarithms but became lost in the algebra. Those who used their GDC to find when the function equalled 0.395 typically did so successfully. A common error for those who obtained a correct value for time in minutes was to treat 5.55 hours as 5 hours and 55 minutes after 13:00.

5c. At 13:00, when there is no medication in Jose's bloodstream, he takes his first dose of medication. He can take his medication again when the concentration of medication reaches 0.395 milligrams per litre. What time will Jose be able to take his medication again? *[5 marks]*

Markscheme

set up equation ***(M1)***

e.g.
$A(t) = 0.395$

attempting to solve ***(M1)***

e.g. graph, use of logs

correct working ***(A1)***

e.g. sketch of intersection,
$0.014t \log 0.5 = \log 0.0395$

$t = 333.00025\ldots$ ***A1***

correct time 18:33 or 18:34 (accept 6:33 or 6:34 but nothing else) ***A1*** ***N3***

[5 marks]

Examiners report

For a later question in Section A, a pleasing number of candidates made good progress. Some candidates believed that raising a base to the zero power gave zero which indicated that they most likely did not begin by analysing the function with their GDC. For part (c), many candidates could set up the equation correctly and had some idea to apply logarithms but became lost in the algebra. Those who used their GDC to find when the function equalled 0.395 typically did so successfully. A common error for those who obtained a correct value for time in minutes was to treat 5.55 hours as 5 hours and 55 minutes after 13:00.

Let
$f(x) = \log_3 \frac{x}{2} + \log_3 16 - \log_3 4$, for
$x > 0$.

6a. Show that
$f(x) = \log_3 2x$. *[2 marks]*

Markscheme

combining 2 terms ***(A1)***

e.g.
$\log_3 8x - \log_3 4$,
$\log_3 \frac{1}{2}x + \log_3 4$

expression which clearly leads to answer given ***A1***

e.g.
$\log_3 \frac{8x}{4}$,
$\log_3 \frac{4x}{2}$

$f(x) = \log_3 2x$ ***AG*** ***N0***

[2 marks]

Examiners report

Few candidates had difficulty with part (a) although it was often communicated using some very sloppy applications of the rules of logarithm, writing
$\frac{\log 16}{\log 4}$ instead of
$\log\left(\frac{16}{4}\right)$.

6b. Find the value of $f(0.5)$ and of $f(4.5)$.

[3 marks]

Markscheme

attempt to substitute either value into *f* ***(M1)***

e.g. $\log_3 1$, $\log_3 9$

$f(0.5) = 0$, $f(4.5) = 2$ ***A1A1 N3***

[3 marks]

Examiners report

Part (b) was generally done well.

6c. The function *f* can also be written in the form $f(x) = \frac{\ln ax}{\ln b}$.

[6 marks]

(i) Write down the value of *a* and of *b* .

(ii) Hence on graph paper, **sketch** the graph of *f* , for $-5 \le x \le 5$, $-5 \le y \le 5$, using a scale of 1 cm to 1 unit on each axis.

(iii) Write down the equation of the asymptote.

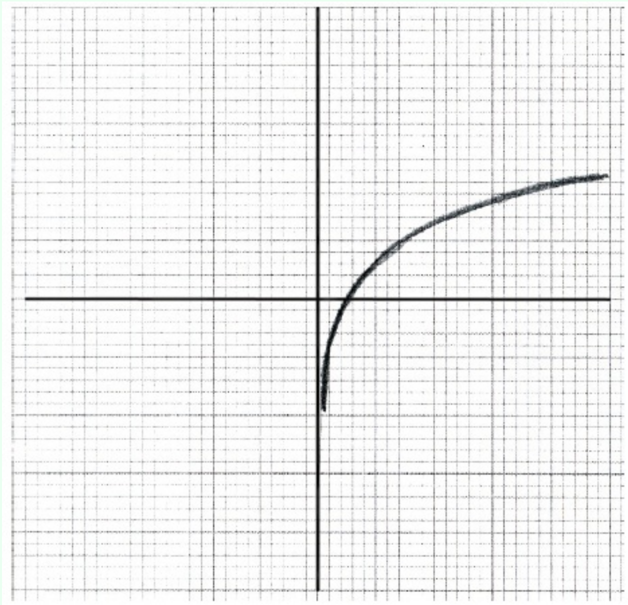
Markscheme

(i)

$a = 2$,

$b = 3$ ***A1A1*** ***N1N1***

(ii)



A1A1A1 ***N3***

Note: Award ***A1*** for sketch approximately through $(0.5 \pm 0.1, 0 \pm 0.1)$, ***A1*** for approximately correct shape, ***A1*** for sketch asymptotic to the *y*-axis.

(iii)

$x = 0$ (must be an equation) ***A1*** ***N1***

[6 marks]

Examiners report

Part (c) (i) was generally done well; candidates seemed quite comfortable changing bases. There were some very good sketches in (c) (ii), but there were also some very poor ones with candidates only considering shape and not the location of the *x*-intercept or the asymptote. A surprising number of candidates did not use the scale required by the question and/or did not use graph paper to sketch the graph. In some cases, it was evident that students simply transposed their graphs from their GDC without any analytical consideration.

6d. Write down the value of $f^{-1}(0)$. *[1 mark]*

Markscheme

$f^{-1}(0) = 0.5$ ***A1*** ***N1***

[1 mark]

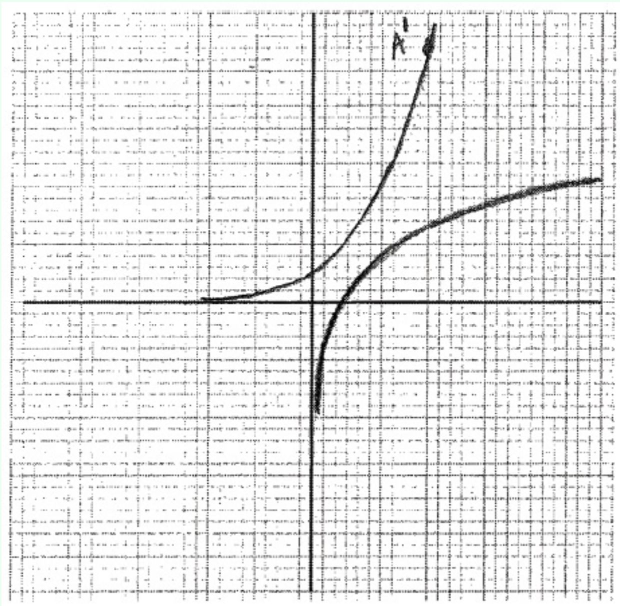
Examiners report

Part (d) was poorly done as candidates did not consider the command term, “write down” and often proceeded to find the inverse function before making the appropriate substitution.

6e. The point A lies on the graph of f. At A, $x = 4.5$. *[4 marks]*

On your diagram, sketch the graph of f^{-1}, noting clearly the image of point A.

Markscheme



A1A1A1A1 ***N4***

Note: Award ***A1*** for sketch approximately through $(0 \pm 0.1, 0.5 \pm 0.1)$, ***A1*** for approximately correct shape of the graph reflected over $y = x$, ***A1*** for sketch asymptotic to x-axis, ***A1*** for point $(2 \pm 0.1, 4.5 \pm 0.1)$ clearly marked and on curve.

[4 marks]

Examiners report

Part (e) eluded a great many candidates as most preferred to attempt to find the inverse analytically rather than simply reflecting the graph of f in the line $y = x$. This graph also suffered from the same sort of problems as the graph in (c) (ii). Some students did not have their curve passing through $(2, 4.5)$ nor did they clearly indicate its position as instructed. This point was often mislabelled on the graph of f. The efforts in this question demonstrated that students often work tenuously from one question to the next, without considering the "big picture", thereby failing to make important links with earlier parts of the question.

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