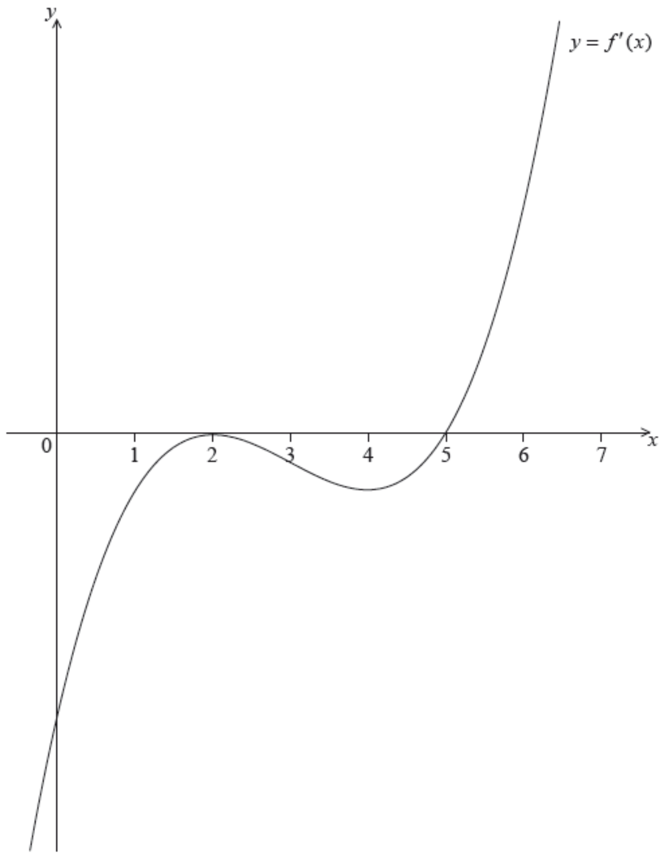
Kinematics and Applications of Derivatives

[216 marks]

Let $y = f(x)$, for
$-0.5 \leq$ x
$\leq$
6.5. The following diagram shows the graph of f', the derivative of f.



The graph of f' has a local maximum when $x = 2$, a local minimum when $x = 4$, and it crosses the
x-axis at the point
$(5,\ 0)$.

1a. Explain why the graph of f has a local minimum when $x = 5$. *[2 marks]*

. . . .

. . . .

. . . .

1b. Find the set of values of x for which the graph of f is concave down. *[2 marks]*

. . . .

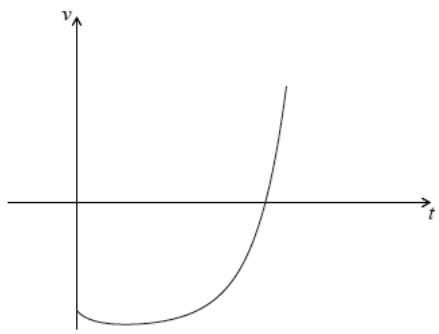
. . . .

. . . .

The velocity $v\ \mathrm{ms}^{-1}$ of a particle after t seconds is given by

$v(t) = (0.3t + 0.1)^t - 4$, for $0 \le t \le 5$

The following diagram shows the graph of v.



2a. Find the value of t when the particle is at rest. *[3 marks]*

. . . .

. . . .

. . . .

2b. Find the value of t when the acceleration of the particle is 0. *[3 marks]*

. . . .

. . . .

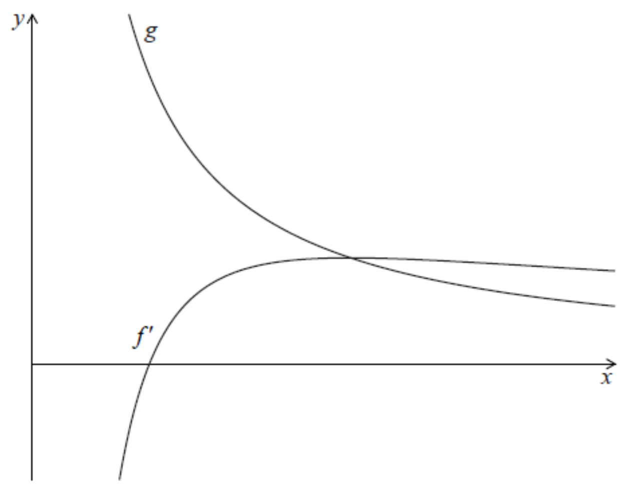
. . . .

Let
$f(x) = \frac{(\ln x)^2}{2}$, for
$x > 0$.

3a. Show that
$f'(x) = \frac{\ln x}{x}$. *[2 marks]*

3b. There is a minimum on the graph of
f. Find the
x-coordinate of this minimum. *[3 marks]*

Let
$g(x) = \frac{1}{x}$. The following diagram shows parts of the graphs of
f' and g.



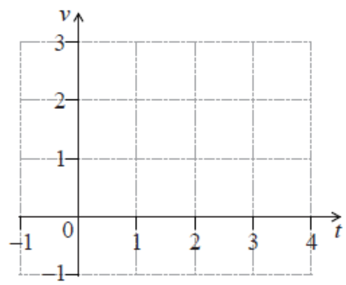
The graph of
f' has an x-intercept at
$x = p$.

3c. Write down the value of
p. *[2 marks]*

3d. The graph of
g intersects the graph of
f' when
$x = q$.
Find the value of
q. *[3 marks]*

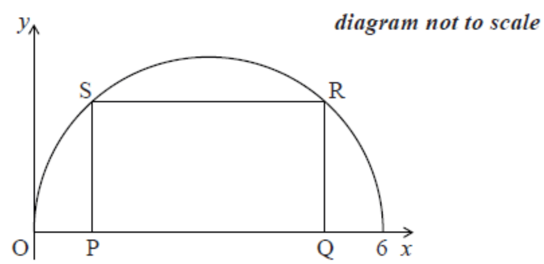
A particle moves along a straight line such that its velocity, $v\ \mathrm{ms}^{-1}$, is given by $v(t) = 10t\mathrm{e}^{-1.7t}$, for $t \geqslant 0$.

4a. On the grid below, sketch the graph of v, for $0 \leqslant t \leqslant 4$. *[3 marks]*



4b. Find the velocity of the particle when its acceleration is zero. *[3 marks]*

Consider the graph of the semicircle given by $f(x) = \sqrt{6x - x^2}$, for $0 \leqslant x \leqslant 6$. A rectangle PQRS is drawn with upper vertices R and S on the graph of f, and PQ on the x-axis, as shown in the following diagram.



5a. Let $\mathrm{OP} = x$.

(i) Find PQ, giving your answer in terms of x.

(ii) Hence, write down an expression for the area of the rectangle, giving your answer in terms of x.

5b. Find the rate of change of area when $x = 2$. *[2 marks]*

5c. The area is decreasing for $a < x < b$. Find the value of a and of b. *[2 marks]*

Consider
$f(x) = \ln(x^4 + 1)$.

6a. Find the value of
$f(0)$. *[2 marks]*

6b. Find the set of values of
x for which
f is increasing. *[5 marks]*

The second derivative is given by
$f''(x) = \frac{4x^2(3-x^4)}{(x^4+1)^2}$.

The equation
$f''(x) = 0$ has only three solutions, when
$x = 0$,
$\pm\sqrt[4]{3}$
$(\pm 1.316\ldots)$.

6c. (i) Find
$f''(1)$. *[5 marks]*

(ii) **Hence**, show that there is no point of inflexion on the graph of
f at
$x = 0$.

6d. There is a point of inflexion on the graph of
f at
$x = \sqrt[4]{3}$
$(x = 1.316\ldots)$. *[3 marks]*

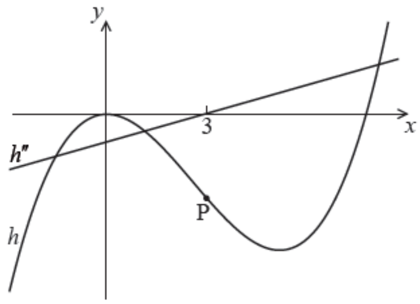
Sketch the graph of
f , for
$x \geq 0$.

Consider the functions
$f(x)$,
$g(x)$ and
$h(x)$. The following table gives some valuesassociated with these functions.

x	2	3
$f(x)$	2	3
$g(x)$	-14	-18
$f'(x)$	1	1
$g'(x)$	-5	-3
$h''(x)$	-6	0

7a. Write down the value of
$g(3)$, of
$f'(3)$, and of
$h''(2)$. *[3 marks]*

The following diagram shows parts of the graphs of
h and
h'' .



There is a point of inflexion on the graph of
h at P, when
$x = 3$.

7b. Explain why P is a point of inflexion. *[2 marks]*

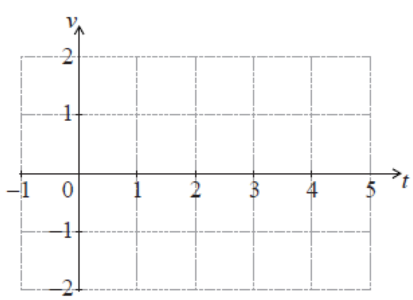
Given that
$h(x) = f(x) \times g(x)$,

7c. find the
y-coordinate of P. *[2 marks]*

7d. find the equation of the normal to the graph of
h at P. *[7 marks]*

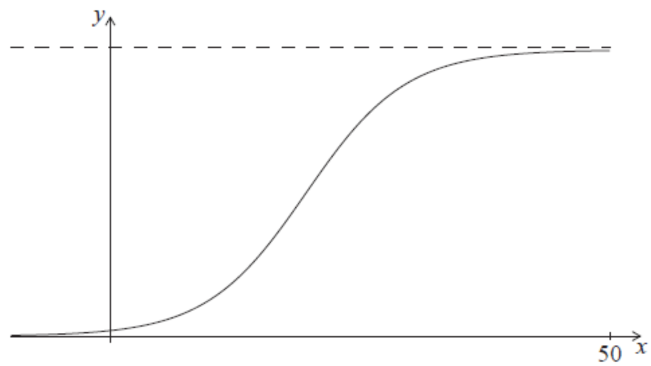
The velocity of a particle in ms^{-1} is given by
$v = e^{\sin t} - 1$, for
$0 \le t \le 5$.

8a. On the grid below, sketch the graph of
v . *[3 marks]*



8b. Find the total distance travelled by the particle in the first five seconds. *[1 mark]*

Let
$f(x) = \frac{100}{(1+50e^{-0.2x})}$. Part of the graph of
f is shown below.



9a. Write down $f(0)$. [1 mark]

9b. Solve $f(x) = 95$. [2 marks]

9c. Find the range of f . [3 marks]

9d. Show that $f'(x) = \frac{1000e^{-0.2x}}{(1+50e^{-0.2x})^2}$. [5 marks]

9e. Find the maximum rate of change of f . [4 marks]

Let
$f(t) = 2t^2 + 7$, where
$t > 0$. The function *v* is obtained when the graph of *f* is transformed by

a stretch by a scale factor of $\frac{1}{3}$ parallel to the *y*-axis,

followed by a translation by the vector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

10. Find $v(t)$, giving your answer in the form $a(t-b)^2 + c$. [4 marks]

The velocity *v* ms^{-1} of a particle at time *t* seconds, is given by
$v = 2t + \cos 2t$, for
$0 \le t \le 2$.

11a. Write down the velocity of the particle when $t = 0$. [1 mark]

11b. When $t = k$, the acceleration is zero. *[8 marks]*

(i) Show that $k = \frac{\pi}{4}$.

(ii) Find the exact velocity when $t = \frac{\pi}{4}$.

11c. When $t < \frac{\pi}{4}$, $\frac{\mathrm{d}v}{\mathrm{d}t} > 0$ and when $t > \frac{\pi}{4}$, $\frac{\mathrm{d}v}{\mathrm{d}t} > 0$. *[4 marks]*

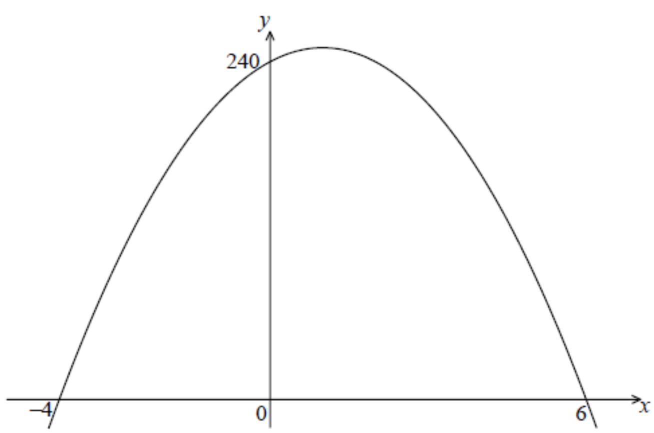
Sketch a graph of *v* against *t* .

Let $g(x) = \frac{\ln x}{x^2}$, for $x > 0$.

12a. Use the quotient rule to show that $g'(x) = \frac{1-2\ln x}{x^3}$. *[4 marks]*

12b. The graph of *g* has a maximum point at A. Find the*x*-coordinate of A. *[3 marks]*

The following diagram shows part of the graph of a quadratic function*f* .



The *x*-intercepts are at $(-4, 0)$ and $(6, 0)$, and the *y*-intercept is at $(0, 240)$.

13a. Write down $f(x)$ in the form $f(x) = -10(x-p)(x-q)$. *[2 marks]*

13b. Find another expression for $f(x)$ in the form $f(x) = -10(x-h)^2 + k$. *[4 marks]*

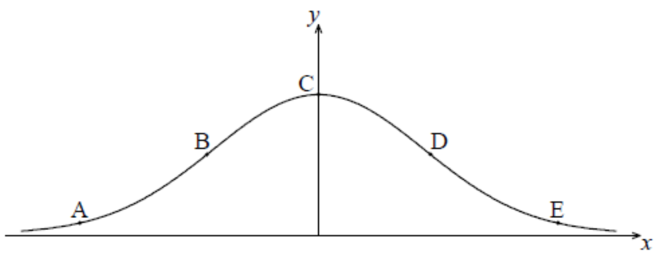
13c. Show that $f(x)$ can also be written in the form $f(x) = 240 + 20x - 10x^2$. *[2 marks]*

13d. A particle moves along a straight line so that its velocity, $v\ \mathrm{ms}^{-1}$, at time t seconds is given by $v = 240 + 20t - 10t^2$, for $0 \le t \le 6$. *[7 marks]*

(i) Find the value of t when the speed of the particle is greatest.

(ii) Find the acceleration of the particle when its speed is zero.

The following diagram shows the graph of $f(x) = \mathrm{e}^{-x^2}$.



The points A, B, C, D and E lie on the graph of f . Two of these are points of inflexion.

14a. Identify the **two** points of inflexion. *[2 marks]*

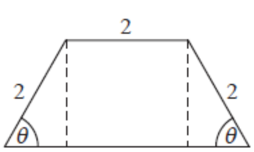
14b. (i) Find $f'(x)$. *[5 marks]*

(ii) Show that $f''(x) = (4x^2 - 2)\mathrm{e}^{-x^2}$.

14c. Find the *x*-coordinate of each point of inflexion. *[4 marks]*

14d. Use the second derivative to show that one of these points is a point of inflexion. *[4 marks]*

The diagram below shows a plan for a window in the shape of a trapezium.



Three sides of the window are $2\ \mathrm{m}$ long. The angle between the sloping sides of thewindow and the base is θ , where $0 < \theta < \frac{\pi}{2}$.

15a. Show that the area of the window is given by $y = 4\sin\theta + 2\sin 2\theta$. *[5 marks]*

15b. Zoe wants a window to have an area of $5\ \mathrm{m}^2$. Find the two possible values of θ . *[4 marks]*

15c. John wants two windows which have the same area A but different values of θ . *[7 marks]*

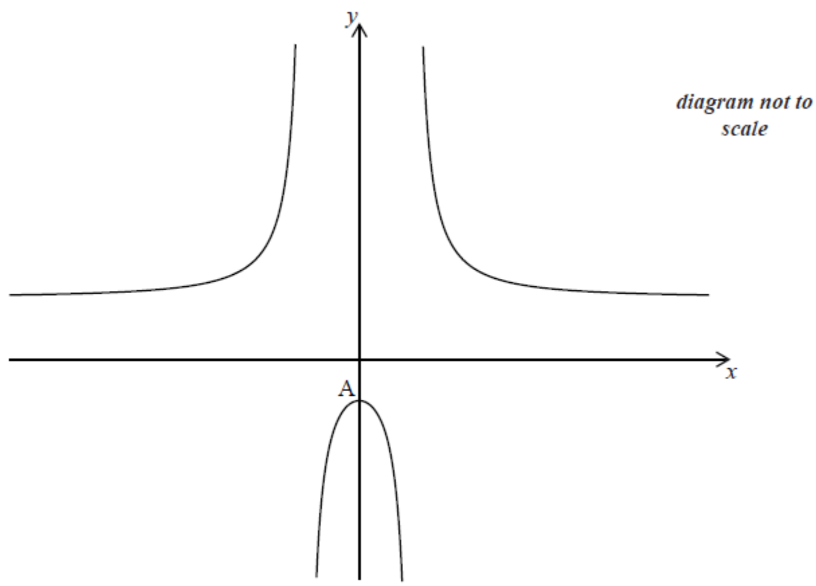
Find all possible values for A .

Consider
$f(x) = x^2 + \frac{p}{x}$,
$x \neq 0$, where *p* is a constant.

16a. Find $f'(x)$. *[2 marks]*

16b. There is a minimum value of $f(x)$ when $x = -2$. Find the value of p . *[4 marks]*

Let
$f(x) = 3 + \frac{20}{x^2-4}$, for
$x \neq \pm 2$. The graph of *f* is given below.



The *y*-intercept is at the point A.

17a. (i) Find the coordinates of A. *[7 marks]*

(ii) Show that $f'(x) = 0$ at A.

17b. The second derivative $f''(x) = \frac{40(3x^2+4)}{(x^2-4)^3}$. Use this to *[6 marks]*

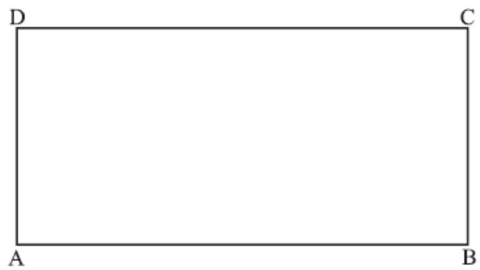
(i) justify that the graph of *f* has a local maximum at A;

(ii) explain why the graph of *f* does **not** have a point of inflexion.

17c. Describe the behaviour of the graph of f for large $|x|$. *[1 mark]*

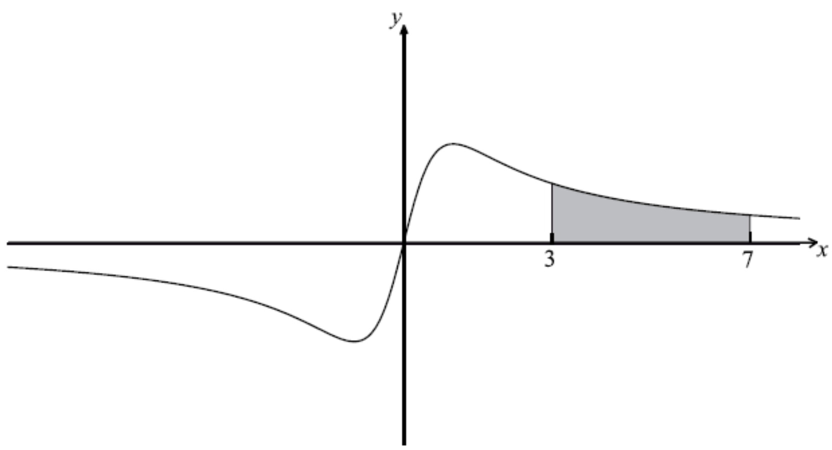
17d. Write down the range of f . *[2 marks]*

A farmer wishes to create a rectangular enclosure, ABCD, of area 525 m^2, as shown below.



18. The fencing used for side AB costs $\$11$ per metre. The fencing for the other three sidescosts $\$3$ per metre. The farmer creates an enclosure so that the cost is a minimum. Find this minimum cost. *[7 marks]*

Let $f(x) = \frac{ax}{x^2+1}$, $-8 \leq x \leq 8$, $a \in \mathbb{R}$ .The graph of *f* is shown below.

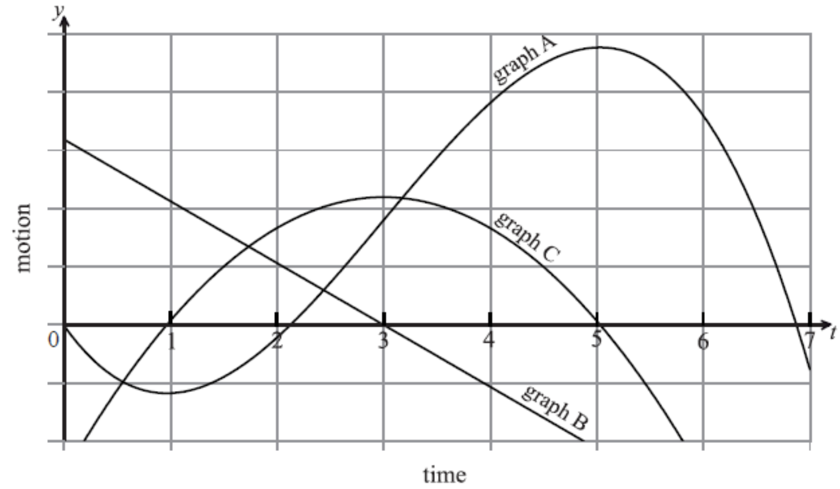


The region between $x = 3$ and $x = 7$ is shaded.

19a. Show that $f(-x) = -f(x)$. *[2 marks]*

19b. Given that $f''(x) = \frac{2ax(x^2-3)}{(x^2+1)^3}$, find the coordinates of all points of inflexion. *[7 marks]*

The following diagram shows the graphs of the **displacement**, **velocity** and **acceleration** of a moving object as functions of time, *t*.

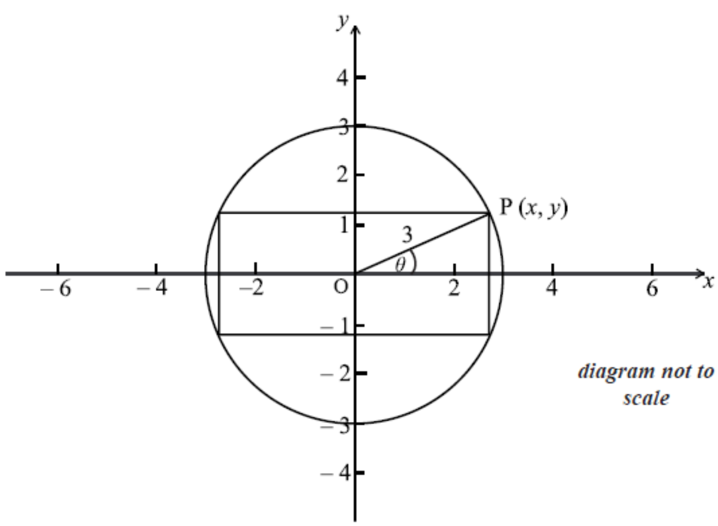


20a. Complete the following table by noting which graph A, B or C corresponds to each function. *[4 marks]*

Function	Graph
displacement	
acceleration	

20b. Write down the value of *t* when the velocity is greatest. *[2 marks]*

A rectangle is inscribed in a circle of radius 3 cm and centre O, as shown below.



The point P(*x* , *y*) is a vertex of the rectangle and also lies on the circle. The angle between (OP) and the *x*-axis is θ radians, where
$0 \le \theta \le \frac{\pi}{2}$.

21a. Write down an expression in terms of θ for *[2 marks]*

(i)
x ;

(ii)
y .

21b. Let the area of the rectangle be A. *[3 marks]*

Show that
$A = 18 \sin 2\theta$.

21c. (i) Find
$\frac{\mathrm{d}A}{\mathrm{d}\theta}$. *[8 marks]*

(ii) Hence, find the exact value of
θ which maximizes the area of the rectangle.

(iii) Use the second derivative to justify that this value of
θ does give a maximum.

In this question *s* represents displacement in metres and *t* represents time in seconds.

The velocity *v* m s^{-1} of a moving body is given by
$v = 40 - at$ where *a* is a non-zero constant.

Trains approaching a station start to slow down when they pass a point P. As a train slows down, its velocity is given by
$v = 40 - at$, where
$t = 0$ at P. The station is 500 m from P.

22a. A train M slows down so that it comes to a stop at the station. *[6 marks]*

(i) Find the time it takes train M to come to a stop, giving your answer in terms of *a*.

(ii) Hence show that
$a = \frac{8}{5}$.

22b. For a different train N, the value of *a* is 4. *[5 marks]*

Show that this train will stop **before** it reaches the station.
